Localization [...] very few believed it at the time, and even fewer saw its importance, among those who failed was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.

P.W. Anderson, Nobel lecture, 1977
<table>
<thead>
<tr>
<th>Year</th>
<th>Authors</th>
<th>Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1958</td>
<td>Anderson</td>
<td>« vanishing of diffusion »</td>
</tr>
<tr>
<td>1960</td>
<td>Mott/Ioffe-Regel</td>
<td>( \ell \leq \frac{\lambda}{2\pi} )</td>
</tr>
<tr>
<td>1965</td>
<td>Mott</td>
<td>Minimum conductivity</td>
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<td></td>
<td></td>
<td>Variable range hopping</td>
</tr>
<tr>
<td>1972</td>
<td>Thouless</td>
<td>Sensitivity to BC: ( g &lt; 1 ) (Thouless criterion)</td>
</tr>
<tr>
<td>1973</td>
<td>Abou-Chacra/Anderson/Thouless</td>
<td>Anderson model on the Caley Tree</td>
</tr>
<tr>
<td>1977</td>
<td>Anderson/Mott</td>
<td>Nobel Prize</td>
</tr>
<tr>
<td>1980</td>
<td>« gang of four »</td>
<td>Scaling theory of open media ( \frac{\partial \log g}{\partial \log L} )</td>
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<tr>
<td>1980</td>
<td>Götze, Vollhardt, Wölfle</td>
<td>Self-consistent transport theory</td>
</tr>
<tr>
<td>1982</td>
<td>Halperin, Pruisken</td>
<td>Scaling theory of Quantum Hall effect</td>
</tr>
<tr>
<td>&gt;1982</td>
<td>Sharvin, Lagendijk, Maret, Maynard,…</td>
<td>Weak localization</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Mesoscopic physics!</td>
</tr>
<tr>
<td>Year</td>
<td>Authors</td>
<td>Title</td>
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<td>-------</td>
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<td>----------------------------------------------------------------------</td>
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<tr>
<td>1983</td>
<td>Fröhlich &amp; Spencer</td>
<td>Mathematical proof for 3D Anderson model</td>
</tr>
<tr>
<td>1984</td>
<td>Anderson</td>
<td>25 years localization « unrecognizable monster »</td>
</tr>
<tr>
<td>1986</td>
<td>Anderson</td>
<td>« Theory of white paint »</td>
</tr>
<tr>
<td>1986</td>
<td>Kramer, Mackinnon, Economou, Soukoulis, Schreiber</td>
<td>Tight binding model and numerical Scaling</td>
</tr>
<tr>
<td>1987</td>
<td>Papanicolaou, Sheng</td>
<td>Prediction of Localization of Seismic Waves in layered Earth Crust</td>
</tr>
<tr>
<td>1987</td>
<td>Souillard</td>
<td>Localization of Gravitational Waves in Universe?</td>
</tr>
<tr>
<td>1988</td>
<td>John</td>
<td>Prediction of Localization of light in Photonic crystals</td>
</tr>
<tr>
<td>1990</td>
<td>Dorokhorov, Mello etal, Beenakker, Altschuler</td>
<td>DMPK equation for wire interactions</td>
</tr>
<tr>
<td>Year</td>
<td>Authors</td>
<td>Results</td>
</tr>
<tr>
<td>------</td>
<td>-------------------------------------------------------------------------</td>
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<tr>
<td>1992</td>
<td>Bell-labs, Weaver, Genack, Exxon (Sheng et al)</td>
<td>2D localization microwaves&lt;br&gt;2D localization of ultrasound&lt;br&gt;Q1D localization microwaves&lt;br&gt;2D localization of bending waves</td>
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<tr>
<td>1991</td>
<td>Lawandy</td>
<td>Observation of laser action in random media&lt;br&gt;(threshold and line narrowing)</td>
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<tr>
<td>&gt;1995</td>
<td>BEC community</td>
<td>Localization of light in BEC cold atomses or the contrary?</td>
</tr>
<tr>
<td>1997</td>
<td>Wiersma, Lagendijk</td>
<td>3D localization of infrared light (contested later)</td>
</tr>
<tr>
<td>&gt;1998</td>
<td>Cao, Wiersma et al. Sebbah, ...</td>
<td>Random lasering from (pre) localized states</td>
</tr>
<tr>
<td>2000</td>
<td>Genack, Beenakker, Van Tiggelen/Lagendijk/Wiersma</td>
<td>Statistics in localized regime (exp)&lt;br&gt;Idem (theo)&lt;br&gt;D(r) in localized regime</td>
</tr>
<tr>
<td>2006</td>
<td>Maret</td>
<td>Anomalous dynamic transmission of light near mobility edge (contested later)</td>
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<tr>
<td>2007</td>
<td>Fishman/Segev</td>
<td>Transverse Light Localization in 2D lattices</td>
</tr>
<tr>
<td>Year</td>
<td>Location</td>
<td>Group/Institution</td>
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<td>------</td>
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<tr>
<td>2008</td>
<td>Delande, Szriftgiser, Garreau</td>
<td></td>
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<tr>
<td>2011</td>
<td>Urbana Champaign Palaiseau group (Aspect)</td>
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<td>VIRGO</td>
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</table>
Yes

- Anderson’s 1958 paper has « often been quoted but hardly ever read » and even less understood
- Several of his results have become mathematical theorems
- Anderson localization has become an « unrecognizable monster »
- Weak localization is (just) a precursor of strong localization
- P.W. Anderson created condensed matter physics

No

- Anderson’s definition is zero diffusion constant
- Anderson shows how wave loops induce localization
- The gang of four show scale dependent diffusion.
- A local defect in a metal induces Anderson localization
- Anderson transition is discontinuous

Maybe

- 90 % of the publications is either wrong, trivial, already shown by Anderson, or not relevant
- Anderson localization would never have been found « accidentally » in a computer simulation
- After this lecture you will finally understand Anderson localization
Mott minimum conductivity

- Thouless criterion and scaling theory
- Quantum Hall effect

MIT and role of interactions
dense point spectrum

Chaos theory (DMPK equation)

Multifractal eigenfunctions

Full statistics of conductance and transmission

Random laser

- Transverse localization
- Anderson tight binding model & Kicked rotor
Diffusion of Waves

\[ \partial_t \rho(\mathbf{r},t) - D \nabla^2 \rho(\mathbf{r},t) \pm \frac{\rho(\mathbf{r},t)}{\tau_{\text{abs.gain}}} = S \delta(t) \delta(\mathbf{r} - \mathbf{r}_s) \]

Diffusion = random walk of waves

\[ \langle \mathbf{r}^2(t) \rangle = \frac{\langle \rho(\mathbf{r},t) \mathbf{r}^2 \rangle}{\langle \rho(\mathbf{r},t) \rangle} = 6D t \]

\[ D = \frac{1}{3} v \ell^* \]

diffusion constant
Multiple Scattering of Waves

Mean free path

« particle language »: mean distance between successive scatterings

« Wave language »  distance to randomise phase

$\langle \exp(i\phi) = 0 \rangle$
1 cm $\text{Al}_2\text{O}_3$: micro-waves
Genack

250 nm $\text{TIO}_2$: visible light
Maret

$4 \text{ mm Al}$: elastic waves
Page

Gallium Phosphide: visible light
Lagendijk

Gallium Arsenide: visible light 1 µm
Infrared light — Wiersma

Optical speckle 300 nm:
Cold atoms
Aspect
What is localization?
Dimension = 1

Localization of « trapped » states
(« low energies » in math proofs)
Localization of « tightly bound » states
(Anderson model)

Localization of « trapped » states
(« low energies » in math proofs)

Dimension = 1

$V(r)$

« sea level »
Localization of « extended states »
\( E > V_{\text{max}} \)
Classical waves, cold atoms

Localization of « tightly bound » states
(Anderson model)

Localization of « trapped » states
(« @ low energies » in math proofs)

\[ V(r) \]

Dimension = 1

« sea level »

\( r \)
Dimension = 3

V(r)

« metal »

« insulator »

Mobility edge

« sea level »

Classical percolation threshold
Anderson Tight Binding Model (80-90)

\[ H = \sum_{nn'} t \langle n|n'\rangle + \sum_n V_n \langle n|n\rangle \]

\[ V_n \in [-W, W] \text{ iid} \]

Disorder \( W/t \)

Sea level

Isolant

Metal

Band edge

Energy \( E/6t \)

Kroha, Wölfle, 1992

\[ H = \sum_{nn'} t |n\rangle \langle n'| + \sum_n V_n |n\rangle \langle n| \]
Average escape probability at large times

\[
\langle |\psi(r, t)|^2 \rangle = \int \frac{d^d k}{(2\pi)^d} |\phi_0(k)|^2 \int dE A(E, k)P_E(r, t)
\]

Distribution of initial velocities
\[mv = \hbar k\]

Spectral function

Probability of quantum diffusion

\[dP_I(t) = \frac{dV|\psi(r = 0, t)|^2}{\# \text{ micro states}}\]

\# micro states = \[\rho(E)dE dV\]
\[dE = \frac{\hbar}{dt}\]

\[\Rightarrow dP_I(t) = dt \frac{P_E(r = 0, t)}{\hbar \rho(E)}\]
What is the probability to have constructive interference in infinite media?

The probability to interfere is given by

$$P_{\text{interference}} = \frac{1}{\hbar \rho(E)} \int_{D t < \ell^2}^{\infty} dt \frac{S_d \Gamma\left(\frac{d}{2}\right)}{2(Dt)^{d/2}}$$

where

- $\rho(E) \sim S_d \frac{k^{d-1}}{hv}$
- $D = \frac{1}{d} \nu \ell$

This probability is represented by the following cases:

- $d < 2$: $\infty$
- $d = 3$: $\frac{1}{(k\ell)^2}$
- $d > 3$: $\frac{\Gamma\left(\frac{d}{2}\right)}{d-2} \left(\frac{1}{(k\ell)^{d-1}}\right)$

→ Something is wrong with diffusion picture at large times in low dimension or at large disorder
→ Weak localization is precursor
→ $d=2$ is critical dimension
→ Velocity cancels
→ Ioffe Regel type criterion in all dimensions $d > 2$
→ Large dimensions: $k\ell \sim 1/d^{1/d}$
How much extra probability returns at the source by constructive interference in open media?

Probability to interfere:

$$\beta(g) = \frac{1}{\hbar \rho(E)} \int_0^\infty dt \frac{\exp(-Dt/L^2)}{L^d} = \frac{1}{\hbar \rho(E) D(E) L^{d-2}} \equiv \frac{1}{g}$$

- Velocity cancels, only length scales $\lambda, \ell, L$
- Extended regime $g > 1$, $d \log g / d \log L > 0$,
  Localized regime $g < 1$, $d \log g / d \log L < 0$
  Mobility edge $g = 1$ is critical point $dg/dL = 0$
- When $d < 2$ always localization, $g < 1$, $dg/dL < 0$
- When $d > 2$ mobility edge at $k_l = 1$, with:

$$D \propto v \frac{\ell^{d-1}}{L^{d-2}}$$
Localization for a physicist

Number of States \( N(E) = L^d \rho(E) dE \sim \frac{L^d k^{d-1}}{\hbar v} dE \),

Thouless energy \( \frac{\hbar D}{L^2} \)

\( g = \frac{\hbar \rho(E) D(E) L^{d-2}}{\rho(E) D(E) L^{d-1}} = \frac{N(E) \delta E_{\text{Thouless}}}{\Delta E_{\text{level}}} = \frac{\delta E_{\text{Thouless}}}{\Delta E_{\text{level}}} \)

Onset of Anderson localization

return proba = 1 \( \iff \) level width = level spacing \( \iff \) \( g = 1 \)

Thouless & Edwards, 1972

Bart van Tiggelen IHP 2019 Simons Collaboration on Localization
Selfconsistent theory of localization

approximate extension of diffusion
→ finite size scaling built in
→ allows engineering
→ Exhibits basic features

\[
- \nabla \cdot D(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')
\]

\[
\frac{1}{D(\mathbf{r})} = \frac{1}{D_B} + \frac{G(\mathbf{r}, \mathbf{r})}{v E \rho(\omega) \ell}
\]

(Vollhardt & Wölfle, 1980)

reciprocity

quantum probability density

quantum RETURN probability density
\[ \left[ -i\Omega - \nabla \cdot \mathcal{D}(\mathbf{r}, \Omega) \nabla \right] C(\mathbf{r}, \mathbf{r}', \Omega) = \delta(\mathbf{r} - \mathbf{r}') \]

Anomalous diffusion

\[ \frac{1}{\mathcal{D}(\mathbf{r}, \Omega)} = \frac{1}{D_B} + \frac{12\pi}{k^2 \ell} C(\mathbf{r}, \mathbf{r}, \Omega) \]

diffusion constant suppressed by return probability

\[ C \equiv z_0 \left[ \mathcal{D}(z, \Omega) / D_B \right] \partial_z C = 0 \]

Boundary condition (no flux in)
Test 1: toy model: Slab with periodic Boundary Conditions

\[ D(r) = D \Rightarrow G(r, r) = \int_{q<1/\ell} d^2q \frac{1}{D} \sum_{n \neq 0} \frac{1}{n^2 \pi^2 / L^2 + q^2} \]

\[ \Rightarrow D(L) \approx D_B \left( 1 - \frac{1}{(k\ell)^2} + \frac{\ell}{L} \right) \]

OK scaling theory
Test 2: numerical: Quasi-1D wave guide

\[ D(x) \equiv -\frac{\langle J_x(x) \rangle}{\langle \partial_x W(x) \rangle} \]

Yamilov, Skipetrov, et al., 2010
Test 3: experiment: Observation of Anderson Localization of Elastic Waves
D = 1.3 m²/s,
absorption = 0
ℓ = 2.5 mm (L=14.5 mm)
ν_E = 1.6 km/s
R = 0.85
kℓ = 1.6

D_B = 16 m²/s
ℓ_B = 2 mm
t_a = 160 µs
ξ = 15 mm
kξ = 1.8
Diffuse: $\langle \rho^2 \rangle = 4Dt$

transition: $\langle \rho^2 \rangle \sim L^2$, not $t^{2/3}$

Localized: $\langle \rho^2 \rangle \sim L\xi$
transverse confinement of ultrasound

\[ T(\rho,t) \equiv T(0,t) \exp \left( -\frac{\rho^2}{w(\rho,t)^2} \right) \]

Transverse Size
\[ w(\rho,t) \]

Transverse Transmission
\[ T(\rho,t)/T(0,t) \]

\[ k\ell \approx 1.82 \]
\[ v_E = 17.4 \text{ km/s} = 3.5 \times v_p \]
\[ \xi = 16.3 \text{ mm} \]

\[ D(t) \times \]
\[ D(L) \times \]
\[ D(r,t-t') \]
Multifractality of wave function

extended  critical  localized

multifractal
**Multifractality of wave function**

Evers and Mirlin (2008).

\[
I_b = \frac{\int_{b^d} d^d r I(r)}{\int_{L^d} d^d r I(r)} \quad \lambda << b << L
\]

\[
P_q = \sum_b (I_b)^q = \left( \frac{L}{b} \right)^{-d(q-1)+\Delta(q)}
\]

\[
P(\log I_b) \propto \left( \frac{L}{b} \right)^f \left( -\frac{\log I_b}{\log L/b} \right)
\]

- **anomalous Inverse Participation Ratio**
- **Lognormal Probability Distribution Function**

\[
\Delta(q) = \gamma q (1-q) \iff f(\alpha) = -\frac{1}{4\gamma} (\alpha - d - \gamma)^2
\]
\[
\Delta_q \approx 0.21 \, q \,(1-q)
\]

\[
\gamma \propto \frac{1}{g \, F(L/\xi(E))}
\]
Anderson mobility gap probed by dynamic coherent backscattering

L. A. Cobus, S. E. Skipetrov, A. Aubry, B. A. van Tiggelen, A. Derode, and J. H. Page

\( D = 0.7 \text{ mm}/\mu\text{s} \)

\( \xi = 16.5 \text{ mm} \)

\( L_1 = 25 \text{ mm} \)

\( L_2 = 38 \text{ mm} \)
1.243 ± 0.007 MHz
1.198 ± 0.001 MHz
Theorem:
For isotropic uniform incidence $\nu=1$

\[
\langle t \rangle = \frac{4\nu V}{v_E S},
\]

Savo, Pierrat, Najar, Carminati, Rotter, Science, November 2017
Time delay in localized media

\[
\frac{W(s = 0)}{F_{in}} = - \left. \frac{dT(s)}{ds} \right|_{s=0} - \left. \frac{dR(s)}{ds} \right|_{s=0}
\]

\[
= \left\langle T(\omega) \frac{d\phi_T(\omega)}{d\omega} \right\rangle + \left\langle R(\omega) \frac{d\phi_R(\omega)}{d\omega} \right\rangle
\]

\[
= \sum_{ab} \int dt \ I_{ab}(t) t / \sum_{ab} \int dt \ I_{ab}(t)
\]

= Total delay time
Time delay in localized media

SC theory: Quasi 1D wave guide, incidence from left
Relative delay in reflection

Relative delay in transmission

Total delay

\[ \langle t \rangle = \frac{3(\tau_s + \tau_0)L}{2v_E} \]
Thank you