Eigenvalues of a Robin Laplacian with a large parameter in the boundary condition

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Based on joint works with Bernard Helffer (Orsay→Nantes), Magda Khalile (Orsay→Hannover), Hynek Kovařík (Brescia), Thomas Ourmières-Bonafos (Orsay→Paris-Dauphine), Nicolas Popoff (Marseille→Bordeaux)
Problem setting

Robin Laplacian

$\Omega \subset \mathbb{R}^d$ suitably regular open set, $d \geq 2$, $\alpha \in \mathbb{R}$

The Robin Laplacian $Q_{\Omega}^\alpha$ in $L^2(\Omega)$ is

$$Q_{\alpha}^\Omega u = -\Delta u \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = \alpha u \text{ on } \partial \Omega.$$
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The associated sesquilinear form $q^\Omega_{\alpha}$ is given by

$$q^\Omega_{\alpha}(u,u) = \int_{\Omega} |\nabla u|^2 \, dx - \alpha \int_{\partial \Omega} |u|^2 \, ds, \quad u \in H^1(\Omega).$$
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By the min-max principle, under suitable assumptions (e.g. bounded Lipschitz) the eigenvalues $E_j(Q^\Omega_\alpha)$ are given by

$$E_j(Q^\Omega_\alpha) = \inf_{S \subset H^1(\Omega), \dim S = j} \sup_{u \in S, u \neq 0} \frac{q^\Omega_\alpha(u,u)}{\|u\|^2_{L^2(\Omega)}}.$$
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The Robin Laplacian $Q_\alpha^\Omega$ in $L^2(\Omega)$ is

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in particular, $E_j \to -\infty$ at fixed $j$ as $\alpha \to +\infty$. 

First eigenvalue
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Lower bound (Sobolev estimate)

Classical result on Sobolev spaces: If $\Omega$ is bounded Lipschitz, then for some $K > 0$ and all $\varepsilon \in (0,1)$ and $u \in H^1(\Omega)$ there holds

$$\int_{\partial \Omega} |u|^2 ds \leq K \left( \varepsilon \int_{\Omega} |\nabla u|^2 dx + \frac{1}{\varepsilon} \int_{\Omega} |u|^2 dx \right)$$
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$$\Rightarrow \quad E_1(Q^\Omega_\alpha) \geq -C\alpha^2, \ C > 0, \ \alpha \to +\infty.$$
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Asymptotics

\[
E_1(Q^\Omega_\alpha) \sim -C_\Omega \alpha^2, \quad C_\Omega \geq 1, \quad -C_\Omega = \inf_{x \in \partial \Omega} E_1(Q^\Lambda_x^1),
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with \( \Lambda_x \) being the tangent cone at \( x \), and \( C_\Omega = 1 \) for smooth \( \Omega \).
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**Domains with peaks (Kovařík–P ’2018)**

If $\Omega$ has a peak of the type $\sqrt{x_1^2 + \cdots + x_{d-1}^2} < x_d^p$ with $1 < p < 2$, then $E_j \sim -\varepsilon_j \alpha^{2-p}$, where $(-\varepsilon_j)$ are the eigenvalues of an explicit 1D operator depending on $d$ and $p$. 
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An “almost separation of variables” near the boundary, essentially the main term is governed by the 1D operator $T_\alpha$ in $L^2(0, +\infty)$ in the normal direction:

$$f \mapsto -f'', \quad -f'(0) = \alpha f(0), \quad \text{spec } T_\alpha = \{-\alpha^2\} \cup [0, +\infty).$$
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$$E_j(\Omega_\alpha) = -\alpha^2 - (d - 1)H_{\max}(\Omega)\alpha + o(\alpha), \quad H_{\max}(\Omega) = \sup \text{ of the mean curvature } H \text{ of } \partial \Omega.$$
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- Various extensions, e.g. eigenvalue counting function, tunneling effect: Helffer–Kachmar–Raymond ’2017, Kachmar–Keraval–Raymond ’2016, ...
Spin-off: a link to isoperimetric inequalities
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$$H_{\text{max}}(B) \leq H_{\text{max}}(\Omega) \quad \text{(H)}.$$ 

Hence, if $B$ maximize $E_1$ in a class of domains $\Omega$, it also satisfies (H) in the same class.

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Now, revenons à nos moutons:
We continue with:

- \( \Omega \subset \mathbb{R}^2 \) polygon with \( V \) vertices \( A_v \),
- side lengths \( l_v := |A_{v+1} - A_v| \),
- half-angles \( \theta_v \) at \( A_v \)

**Question:** Given \( j \in \mathbb{N} \), what is the behavior of \( E_j(Q^\Omega_\alpha) \) as \( \alpha \to +\infty \)?
Non-smooth domains: beyond the first eigenvalue

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**Theorem (Khalile–P ’2018):** $\sigma_{ess}(T_\theta) = [-1, +\infty)$,
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**Theorem (Khalile–P ’2018):** $\sigma_{ess}(T_\theta) = [-1, +\infty)$, the spectrum in $(-\infty, -1)$ consists of $K_\theta < \infty$ eigenvalues,
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**Theorem (Khalile–P '2018):** \( \sigma_{\text{ess}}(T_\theta) = [-1, +\infty) \), the spectrum in \( (-\infty, -1) \) consists of \( K_\theta < \infty \) eigenvalues, which are increasing in \( \theta \), and
Non-smooth domains: beyond the first eigenvalue

We continue with:

- $\Omega \subset \mathbb{R}^2$ polygon with $V$ vertices $A_v$,
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$$K_\theta = 0 \text{ for } \theta \geq \frac{\pi}{2}, \quad K_\theta \geq 1 \text{ for } \theta < \frac{\pi}{2} \quad \text{and} \quad E_1 = -\frac{1}{\sin^2 \theta},$$

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and the first \( K_\theta \) eigenfunctions are exponentially localized near the vertex.
Corner-induced eigenvalues

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- For $j \leq K = K_{\theta_1} + \cdots + K_{\theta_V}$ one has

$$E_j(Q_\alpha^\Omega) = E_j(T_{\theta_1} \oplus \cdots \oplus T_{\theta_V}) \alpha^2 + O(e^{-c\alpha}), \quad c > 0, \quad \text{as } \alpha \to +\infty.$$
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- The associated eigenfunctions are exponentially localized near the corners and are close (in a rigorously defined sense) to linear combinations of the (suitably truncated and rotated) eigenfunctions of \( T_{\theta_v} \) corresponding to the convex corners (i.e. with \( \theta_v < \frac{\pi}{2} \)).
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If there are two or more equal angles, one may be interested in the tunneling effect between them (which leads to calculating the exponentially small remainder in a more precise way). Up to know, only a very particular configuration with two corners was studied (Helffer–P ’2015) + some explicit computations (separation of variables) for rectangles and the equilateral triangle (McCartin’2011).
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![Diagram](image)

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Adaptation to the Robin case?
Edge-induced eigenvalues (2)

One can further decompose $\Omega_\delta$ into corner neighborhoods $U_{v,\delta}$ and (slightly shortened) edge neighborhoods $W_{v,\delta}$, separated by segments $\Sigma_{v,\delta}^\pm$:

$$U := \bigcup U_{v,\delta},$$
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First idea: by imposing Dirichlet/Neumann at $\Sigma$ one arrives at a direct sum of operators in $U$ and $W$, but: no reasonable bound for the operators in the convex parts of $U$!
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**Definition:** A corner of opening angle $\theta \in \left(0, \frac{\pi}{2}\right)$ is *non-resonant* if the Laplacian $L_{\theta,R}$ in the truncated sector (see Figure), with the Robin boundary condition $\partial_n u = u$ at $OA_R^{\pm}$ and the Neumann one on the other sides, satisfies $E_{K_\theta + 1}(L_{\theta,R}) \geq -1 + CR^{-2}$ with some $C > 0$ as $R \to +\infty$. 
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Observation (separation of variables + a kind of monotonicity): all $\theta \geq \frac{\pi}{4}$ are non-resonant.
Edge-induced eigenvalues (3): main result
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**Theorem (Khalile, Ourmieres-Bonafos, P ’2018):**
If all corners of $\Omega$ are concave or non-resonant (in particular, if $\theta_v \geq \frac{\pi}{4}$ for all $v$), then, for any fixed $j$,

$$E_{K+j}(Q_\alpha^\Omega) = -\alpha^2 + E_j(D_1 \oplus \cdots \oplus D_V) + O\left(\frac{\log \alpha}{\sqrt{\alpha}}\right),$$

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Remark:
We do not expect our result to be optimal.
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We do not expect our result to be optimal. Nevertheless, the above result is not valid without any additional assumption on the corners:

- for the equilateral triangle ($\theta_v \equiv \frac{\pi}{6}$), the separation of variables (McCartin ’2011) gives $K = 3$ (i.e. 3 corner-induced eigenvalues) and $E_{3+j}(Q^{\Omega}_{\alpha}) = -\alpha^2 + E_j(L) + o(1)$ with $L$ the Laplacian on $\partial \Omega$ (no obstacle at the corners).
- In particular, $E_4(Q^{\Omega}_{\alpha}) = -\alpha^2 + o(1)$, while $E_1(\oplus D_v) > 0$. 

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- In particular, $E_4(Q^\Omega_\alpha) = -\alpha^2 + o(1)$, while $E_1(\oplus D_v) > 0$.

**Still under study:** An analogous result is expected for curvilinear polygons $\Omega$ with concave or non-resonant corners:

$$E_{K+j}(Q^\Omega_\alpha) = -\alpha^2 + E_j(L^D_\alpha) + r(\alpha),$$

$$L^D_\alpha = -\partial^2 - \alpha H$$ on $\partial \Omega$ with the Dirichlet boundary condition at the corners,

but with a complicated remainder $r(\alpha)$. 


Edge-induced eigenvalues: scheme of the proof (1)
We work in $\Omega_\delta$ with $\delta = c_0 \frac{\log \alpha}{\alpha}$ with $c_0 > 0$ chosen sufficiently large.
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For the upper bound it is sufficient to impose Dirichlet boundary condition at all artificial boundaries.
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For the upper bound it is sufficient to impose Dirichlet boundary condition at all artificial boundaries.

For the lower bound, the following lemma appears to be very useful:

**Lemma (Exner–Post ’2003):** Let $Q$ and $Q'$ be self-adjoint, non-negative, with compact resolvents in $\mathcal{H}$ and $\mathcal{H}'$ and generated by sesquilinear forms $q$ and $q'$. Let $j \in \mathbb{N}$ and assume that there exists $J : D(q) \to D(q')$ such that, for some $\varepsilon_1 \leq (1 + E_j(Q))^{-1}$ and $\varepsilon_2 > 0$,

\[
\|u\|^2 - \|Ju\|^2 \leq \varepsilon_1 (q(u,u) + \|u\|^2),
\]

\[
q'(Ju, Ju) - q(u,u) \leq \varepsilon_2 (q(u,u) + \|u\|^2),
\]

then

\[
E_j(Q') \leq E_j(Q) + \frac{(E_j(Q)\varepsilon_1 + \varepsilon_2)(1 + E_j(Q))}{1 - (1 + E_j(Q))\varepsilon_1}
\]
Edge-induced eigenvalues: scheme of the proof (2)

One obtains the lower bound for the operator $B$ in $L^2(\Omega_\delta)$ given by the form

$$b(u,u) = \int_{\Omega_\delta} |\nabla u|^2 dx - \alpha \int_{\partial \Omega} |u|^2 ds, \quad u \in H^1(\Omega_\delta), \quad b(u,u) = b^U(u,u) + b^W(u,u).$$

One denotes $\Lambda \subset L^2(\Omega_\delta)$ the subspace spanned by the first $K$ eigenfunctions of $B$. 
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One denotes $\Lambda \subset L^2(\Omega_\delta)$ the subspace spanned by the first $K$ eigenfunctions of $B$. Furthermore, let $L$ the 1D operator $f \mapsto -f''$ on $(0,\delta)$ with $f'(0) + \alpha f(0) = f'(\delta) = 0$, then one denotes $(E,\Psi)$ its first eigenpair with a normalized $\Psi$. (I.e. $L$ is action of $B$ with respect to the normal coordinate.)
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One uses the preceding lemma with

\[ \mathcal{H} = L^2(\Omega_\delta) \ominus \Lambda, \quad Q := B - E, \quad \mathcal{H}' = \oplus L^2(\lambda_v \delta, \ell_v - \lambda_{v+1} \delta), \quad Q' = \oplus D_v \]
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and the identification operator $J$ by

$$(Ju)(s) = (Pu)(s) - P(\ell_v \delta)\rho(s) - (Pu)(\ell_v - \lambda_v \delta, \ell_v - \lambda_{v+1} \delta)\rho(\lambda_v, \ell_v - \lambda_{v+1} \delta - s),$$

$$(Pu)(s) = \int_0^\delta u(s, t)\Psi(t) \, dt, \quad \rho \text{ is a cut-off function supported near 0}$$

(an adaptation of a construction by Post '2005 for a special class of waveguide junctions)
Edge-induced eigenvalues: scheme of the proof (3)

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The two main technical ingredients are:

- Helffer-Sjöstrand-type estimates for distances between various subspaces (based on Agmon-type decay estimates), as there are several “almost orthogonal” subspaces in play,

- The non-resonance condition, which allows to show that the terms with \(\rho\) in the expression for \(Ju\) are small in a suitable sense. It appears through the control

\[\int_{\Sigma} |u|^2 ds \leq C\delta^2 a\left(b^U(u,u) - E\|u\|^2_{L^2(U)} + \|u\|^2_{L^2(\Omega_\delta)}\right),\]

which is trivial for concave corners, but requires some work for convex ones.
Concluding remarks
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Our definition of “non-resonance corners” is a naive adaptation of a sufficient condition for the absence of threshold resonances in waveguide junctions. The latter condition can be reformulated in several equivalent forms (e.g. non-existence of non-trivial bounded solutions to some problem, a condition for the scattering matrix at the threshold). Our condition is applicable for an explicit range of corner opening.
Concluding remarks

- Our definition of “non-resonance corners” is a naive adaptation of a sufficient condition for the absence of threshold resonances in waveguide junctions. The latter condition can be reformulated in several equivalent forms (e.g. non existence of non-trivial bounded solutions to some problem, a condition for the scattering matrix at the threshold). Our condition is applicable for an explicit range of corner opening.

- A “correct” definition of (non-)resonance corners should probably make use of (suitably defined) scattering matrices associated with sectors. We are not aware of suitable results (any comment would be welcome!). Expectation: if \( \theta \to K_\theta \) is constant near \( \theta = \theta_0 \), then \( \theta_0 \) is non-resonant.