Consequences of Coherence in Wave Propagation in Random Media

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Outline

Particle diffusion

→ Coherent wave transport

*Statistics of transport*

*Eigenstates (resonances, quasi-normal modes)*

*Transmission eigenchannels*

*Control energy transmitted through and inside random medium*

Universality in wave propagation in random media from

*Statistics of propagation at all length scales in 1D*

*Ballistic to localized wave*

*1D to multichannel systems*
Brownian Motion

\[ \langle R^2 \rangle = 2dDt \]
Wave Scattering and Transport in Random Media

Ohm’s Law
\[ V = IR \]
\[ I = V/R = VG \]
\[ G = A\sigma/L \]

Dimensionless conductance, \( g = G/(e^2/h) = N\ell/L = T, \ N \sim A/\lambda^2 \)

Transmittance, \( T \)
Waves in Mesoscopic Materials

The wave is scattered elastically many times within the sample and forms a stable random speckle pattern within the sample of energy or particle density as a result of interference of the temporally coherent wave.

In electronics, the wavefunction can be multiply scattered and temporally coherent when it is on a mesoscopic scale—intermediate between the microscopic atomic scale and the macroscopic domain. This can be achieved on micron-sized samples at ultralow temperatures.

For classical waves, the wavelength and so the size of scattering elements are much greater than the atomic scale and so the sample is static and mesoscopic.
Waves in Mesoscopic Materials

Averaging intensity over random sample configurations washes out the speckle pattern. In the limit of large transmittance, the smooth spatial and temporal distribution found is given by solving the diffusion equation.

But the influence of wave interference still survives averaging in a host of mesoscopic phenomena for ballistic, diffusive and localized waves.

Anderson localization

Single Parameter Scaling

It is not enough to describe the scaling of the conductance, $g$. For localized waves, the conductance does not self-average and fluctuations are large.

Must describe the full distribution of conductance

In a given dimension, the full distribution should depend upon a single dimensionless parameter, the Thouless number:

$$\delta = \frac{\delta\nu}{\Delta\nu} = g$$

Note, however, that in 1D, $\ln T$ self averages and so the distribution $P(\ln T)$ should be a Gaussian. So the variance and centroid of the distribution should be related:

$$\text{var}(\ln T) = -2\langle\ln T\rangle = -2L / \ell$$

SHORT COMMUNICATIONS

WAVEGUIDES WITH RANDOM INHOMOGENEITIES AND BROWNIAN MOTION IN THE LOBACHEVSKY PLANE

M. E. GERTSENSHITEIN AND V. B. VASIL’EV

1. Introduction and Statement of the Problem

In this work we will show how, with the help of Lobachevsky geometry one can obtain a precise solution of the radio-engineering problem of a statistically inhomogeneous waveguide—of a waveguide (transmission line) with random inhomogeneities.

(Summary)

It has been shown that the probability density for the continuous random process of the resultant of independent values which are summed up according to the linear-fractional law satisfies the diffusion equation in the Lobachevsky plane. Green’s function of the diffusion equation, which apparently is a new distribution, has been found.

Reflectivity: \( r = |r| e^{i\phi} \), \( \eta = \frac{1}{2} \ln \frac{1+|r|}{1-|r|} \), \( s = L/\ell \)

\[
P(\eta, s) \rightarrow P(T, s)
\]
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Waves in Mesoscopic Materials

In the limit in which the transmittance is large, averaging intensity over random sample configurations washes out the speckle pattern and gives the smooth spatial and temporal distribution found by solving the diffusion equation.

But the influence of wave interference still survives averaging in a host of mesoscopic phenomena for ballistic, diffusive and localized waves.

- Anderson localization
- Universal conductance and transmission fluctuations
- Correlation of intensity and focusing through random media
- Correlation in eigenvalues of transmission matrix (TM)
- Profiles of eigenchannels of TM
- Modal excitation and selectivity
- Scaling of energy density in random media & topological insulators (TIs)
Measurement of Microwave Speckle Patterns

Vector Network Analysis

Amplifier

Superposition of modes:

\[ E_j(x, y, \omega) = \sum_n a_{n,j}(x, y) \frac{\Gamma_n / 2}{\Gamma_n / 2 + i(\omega - \omega_n)} \]
Field Spectra and Modes

Central Frequency: 10.1502 GHz
Width: 2.7 MHz

Central Frequency: 10.1478 GHz
Width: 5.1 MHz

Central Frequency: 10.1502 GHz
Width: 2.7 MHz
Field Spectra and Modes

10.15 GHz
Dynamics of Localized Waves

Transmission Matrix: Eigenchannels and Eigenvalues

Transmission of $>0.50$ in selected channel in sample with $<T>\sim 0.1$

Find statistics of transmission eigenvalues $\tau_n$

Popoff, Lerosey, Carminati, Fink, Boccara, Gigan, PRL 104 (2010)
Shi, Genack, PRL 108 (2012)
Transmission Matrix

\[ E_{a}^{+} \rightarrow S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \rightarrow E_{b}^{+} \]

Transmission matrix \( t \)  
\[ E_{out} = tE_{in} \quad E_{b}^{+} = \sum_{a=1}^{N} t_{ba} E_{a}^{+} \]

Singular Value Decomposition:  
\[ t = U \Lambda V^{\dagger} = \sum_{n=1}^{N} u_{n} \lambda_{n} v_{n}^{\dagger} \]

Transmittance:  
\[ T = \sum_{ab}^{N} |t_{ba}|^{2} = Tr(tt^{\dagger}) = \sum_{n}^{N} \lambda_{n}^{2} = \sum_{n}^{N} \tau_{n} \]

Transmission Eigenvalues

\[ \tau_n = \cosh^{-2} \frac{L}{\xi_n} = \cosh^{-2} x_n \]

\[ x_n = \frac{L}{\xi_n} = \frac{(2n-1)L}{2\xi}, \quad \xi = N\ell, \quad n < N / 2 \]

\[ \Delta x = x_{n+1} - x_n = \frac{L}{\xi} = \frac{L}{N\ell} = 1 / g_0 \]

\[ \tau_n \text{ varies from 1 to } \sim \exp(-2L/\ell) \]

Dorokhov, JETP Lett. 36, 318 (1982)
Transmission Eigenvalues

(a)
Transmission Eigenvalues

(a) $g = 6.9$

(b) $g = 0.37$

$P(\ln \tau_n)$ vs $\ln \tau_n$

$P(\ln \tau)$ vs Channel index $n$
Modal Excitation

(a)

(b)

Random wavefront

Transmission

Frequency (GHz)
Selective Modal Excitation: Modal Transmission Matrices

\[ t_{ba}(\omega) = \sum_n \frac{t_{ba}^n}{\omega - \omega_n + i\Gamma_n/2} = \sum_n t_{ba}^n \varphi_n(\omega) \]

- Energy in selected mode enhanced by factor \( N \)
- Correlation in speckle patterns of neighboring modes limits modal selectivity

Simulations of Energy Density Profiles of Transmission Eigenvalues

\[ \tau = 1, 0.5, 0.1, 0.001 \] for a diffusive sample with \( L/\xi = 0.05 \). Single configuration

Davy, Shi, Park, Tian, Genack, Nat. Commun. 6 (2015)
Choi, Mosk, Park, Choi, PRB 83 (2011)
Eigenchannel Profiles of Partially Transmitting Channels

\[ W_\tau(x) = S_\tau(x)W_{\tau=1}(x) \]

\[ S_\tau(x) \] must satisfy conditions:

\[ S_\tau(x = L) = \tau \quad S_\tau(x = 0) = 1 + (1 - \tau) = 2 - \tau \]

\[ \tau = \cosh^{-2}\left(\frac{L}{\xi'}\right) \quad \text{Guess:} \]

\[ S_\tau(x) = 2 \frac{\cosh^2 \left( \left( \frac{L-x}{\xi'} \right) \right)}{\cosh^2 \left( \frac{L}{\xi'} \right)} - \tau \]

\[ S_\tau(x) = 2 \frac{\cosh^2 \left( \frac{h(x/L)(L-x)/\xi'}{\xi'} \right)}{\cosh^2 \left( \frac{h(x/L)L}{\xi'} \right)} - \tau \]

\[ S_\tau(x) \] independent of \( \tau \) and \( L \) for \( \ell < L < \xi \)
Does the Diffusion Approximation Work?

\[ G(z, 0) = \delta(z - a) = \sum_{m} \sin \left( \frac{m \pi a}{L} \right) \sin \left( \frac{m \pi z}{L} \right) = \sum_{m} a_m \sin \left( \frac{m \pi z}{L} \right), \text{with } a_m = \sin \left( \frac{m \pi a}{L} \right) \]

\[ G(z, t) = \sum_{m} \sin \left( \frac{m \pi a}{L} \right) \sin \left( \frac{m \pi z}{L} \right) \exp(-D\left(\frac{m \pi}{L}\right)^2 t) \]

Measurement
Diffusion theory

Yoo, Liu Alfano, PRL 64 (1990)
Does the Diffusion Approximation Work?

Yoo, Liu Alfano, PRL 64 (1990)
Zhang, Jones, Schriemer, Page, Weitz, Sheng, PRE 60 (1999)
Kop, De Vries, Sprik, Lagendijk, PRL 79 (1997)
Eigenchannel Profiles for Ballistic Waves

Puzzling early measurements:
- $T(L)$ diffusive even for $L \approx 2\ell$
- Delay time, $\tau_D$, nearly ballistic up to $L \approx 5\ell$

Yoo, Liu Alfano, PRL 64 (1990)
Kop, De Vries, Sprik, Lagendijk, PRL 79 (1997)
Zhang, Jones, Schriemer, Page, Weitz, Sheng,. PRE 60 (1999)

Consider $W_\tau(x)$, holds key to static and dynamic transport

Paradoxically: it is precisely the diffusive form of steady-state ballistic propagation that leads to slow approach to quadratic diffusive scaling of $\tau_D$

Similarities in steady state transport due to universal statistics of transmission eigenvalues
Measurement of Total Transmission

Thin sample
0.33% polystyrene spheres
0.17 μm diameter
\( \Theta_{\text{wedge}} = 0.86 \text{ deg} \)
Measurement of Total Transmission

0.33% polystyrene spheres
0.17 μm diameter

Θ_wedge = 5.88 deg

L from 0.012λ to 1.5λ

z_b = 0.92 mm

λ = 1.6 mm
Diffusion in Translucent Media

Diffusion theory:

\[
\langle T_a \rangle_a = T / N = \frac{z_p + z_b}{L + 2z_b}
\]

- \( z_b \): Boundary extrapolation length
- \( z_p \): Penetration depth

\[
z_b = \frac{\ell v}{2v_+} = z_p
\]

\[
\frac{1}{\langle T_a \rangle_a} = \frac{L + 2z_b}{z_p + z_b} \Rightarrow 0 \text{ at } L = -2z_b
\]

Find: \( z_b = 19.2\pm0.1 \mu m \)

Diffusion in Translucent Media

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Transmission Eigenchannels

\[ \tau_n = \cosh^{-2} \frac{L}{\xi_n} = \cosh^{-2} x_n \]

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\[ s = (x_{n+1} - x_n) / \Delta x \]
Profile of Completely Transmitting Eigenchannel

\[ h(x) = 1, \quad S_\tau(x) = 2 \frac{\cosh^2((L - x) / \xi')}{\cosh^2(L / \xi')} - \tau \]

\[ L = 50, \quad s = L / \ell = 0.18 \]
Energy Density Profiles in 1D

\[ s = \frac{L}{\ell} = 0.5 \]
Dwell Time

\[ t_n = \frac{d \theta_n}{d \omega} \sim \int_0^L W_n(x) dx \]

\[ t_n / t_B = \frac{\int_0^L W_n(x) dx}{L} \]

\[ t_D = \sum_{n=1}^{N} \frac{\tau_n t_n}{\sum_{n=1}^{N} \tau_n} \]

Energy Density Profiles and the Scaling of Dynamics

\[ W_\tau(x) = S_\tau(x) W_{\tau=1}(x) \]

For \( L > \ell \), \( S_\tau(x/L) \) is largely independent of \( L \)

\[ \Rightarrow t_D \sim \int_0^L W_1(x) dx \]

\[ W_1(x/L) = 1 + F_1(x/L) = \]

\[ [1 + \frac{v_+}{2v}(L/\ell)][4(x/L)(1-x/L)] \]

\[ \frac{v_+}{2v} = 0.35 \]

\( W_1 \) is the sum of ballistic, diffusive and localization terms

Linear diffusive coefficient \( a = 0.35 \)

Small value of \( v_+/2v \)

→ linear increase in \( t_1 \) until \( L/\ell > 1/0.35 \sim 3 \)
Conclusion

Properties of static and dynamic propagation emerge from characteristics of modes and transmission eigenchannels

- Universal and nonuniversal aspects of transmission in 1D
- Modal correlation and its Impact on modal correlation, pulsed transmission and modal selectivity
- Profiles and dynamics of transmission eigenchannels for diffusive and ballistic waves in Q1D and 1D
- Universal statistics of transmission eigenchannels