GEOMETRY AND THE DIRICHLET PROBLEM IN ANY CO-DIMENSION

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Overview

- Harmonic Measure: where does a random walk first exit a domain?
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**Figure:** A random walk exiting a domain (figure credit Matthew Badger)
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• Dirichlet problem: equilibrium after diffusion.
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• Surprising: if nothing is hidden, the domain is nice.
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- Dirichlet problem: equilibrium after diffusion.
- Not surprising: a nasty domain can have “hidden” parts of the boundary
- Surprising: if nothing is hidden, the domain is nice.
- Very surprising (and recent): higher co-dimension analogues.
Walker goes to each neighbor with equal probability.

**Figure:** Each neighbor has probability 1/4
Walker goes to each neighbor with equal probability.

**Figure:** We keep going until we hit the boundary
Walker goes to each neighbor with equal probability.

**Figure:** Backtracking is allowed
Walker goes to each neighbor with equal probability.

**Figure:** Stop when we hit the boundary
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Walker goes to each neighbor with equal probability.  
Hitting Measure: probability RW hits that part of the boundary.

**Figure:** Hitting Measure of Green starting at Red $\equiv \omega^{\text{Pole}}(\text{Target})$
Random Walk and Hitting Measure

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**Figure:** Hitting Measure Depends on the Pole!
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Hard to compute!!! (Solve 10 eqns with 10 unknowns)
Average of neighbors!
Mean Value Property

\[ u(\text{Point}) = \frac{1}{\# \text{Neighbors}} \sum_{\text{Neighbors}} u(\text{Neighbor}). \]
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Example:

**Figure:** Each red value is the average of the neighboring values
Discrete Harmonic Functions

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Uniqueness: Maximum principle!
Recall:

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**Figure:** How do we fill in the boundary?
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This is the Dirichlet problem.
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**Figure:** How do we fill in the boundary?

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\[ u(\text{Point}) = \sum_{\text{BoundaryPoints}} u(\text{BoundaryPoint}) \omega^{\text{Point}}(\text{BoundaryPoint}). \]
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Expected value!
Harmonic Functions in the Continuum

Mean Value Property: for all $X \in \mathbb{R}^n$ and $R > 0$,

$$\frac{1}{|B(X, R)|} \int_{B(X, R)} u(Y) dY = u(X).$$
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$u$ satisfies mean value property $\iff \Delta u = 0 \iff \sum_{i=1}^n \partial_{x_i x_i}^2 u = 0$. 
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**EX:** $u(x, y) \equiv x^2 - y^2$

**Figure:** Harmonic functions don’t have local extrema credit: laussy.org
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Represents Equilibrium after diffusion.
$X \in \Omega \subset \mathbb{R}^n. \hspace{1em} E \subset \partial \Omega. \hspace{1em} \omega^X(E) = \text{Probability a B.M. exits } \Omega \text{ first in } E.$
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$$\begin{align*}
(D) = \left\{ \begin{array}{ll}
\Delta u_f = 0 & x \in \Omega \\
u_f(Q) = f(Q) & Q \in \partial \Omega 
\end{array} \right. 
\end{align*}$$

**Figure:** Brownian Motion exiting a domain (figure credit Matthew Badger)
Harmonic Measure in the Continuum

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$$u_f(X) = \int_{\partial \Omega} f d\omega^X.$$

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What is the temperature in the interior, given the temperature on the edge?
“Bad” geometry: $\omega$ doesn’t “see” sets of large length.
When does $\omega^X$ not see large sets?

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- Connectivity.

**Figure:** $\omega^{\text{Pole}}$ cannot see the other component
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**Figure:** Brownian motion cannot go down hallways

- Cusps

**Figure:** Brownian motion cannot get to the cusp
So what if $\omega^X$ doesn’t see large sets?

(Quantitative) Well posedness of Dirichlet problem:

$$(D) = \begin{cases} 
\Delta u_f = 0 & x \in \Omega \\
\left. u_f(Q) \right|_{Q \in \partial\Omega} = f(Q) & Q \in \partial\Omega
\end{cases}$$

Figure: Changing data on the cusp doesn’t change the solution

"Theorem": $D$ is (quantitatively) well posed iff $k^X$ is not "too large or too small too often." Call this $A_\infty$.
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Three Examples: $\omega$ vs Length

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A disk, Lipschitz domain and Snowflake (figure from Matthew Badger)
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**Three Examples: ω vs Length**

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For a Lipschitz domain, if \( \omega^0(E) = k\sigma \) and \( k \) is not too small or too big too often.
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For the snowflake \( \omega^0 = k \sigma \) and \( k = +\infty \) or 0 at every point.
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They aren’t the only problem! Fractals!

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Lots of work: Ahlfors, Bishop, Carleson, David, E., Fabes, Garnett, Hofmann, Jerison, Kenig, Laurentiev, Mayboroda, Nyström, Øksendal, Pipher, Riesz (x2), Salsa, Toro, Uriarte-Tuero, Volberg, Wolff, Zhao...Many More.
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When $n = 2$: connections to the complex analysis.
Associated with Harmonic measure $\omega^X$ is Green function $G(X, -)$.
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\begin{cases}
G(X, Y) > 0 & X \neq Y \in \Omega \\
G(X, Q) = 0 & Q \in \partial \Omega \\
\Delta_Y G(X, Y) = \delta_X(Y) & Y \in \Omega.
\end{cases}
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**Figure:** George Green (figure credit wikipedia)
GREEN FUNCTION

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Probability: $G(X, Y) =$ how likely does B.M. go from $X$ to $Y$ without leaving $\Omega$ (tricky!)

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Really hard to compute! Known only for a few domains (half plane, disc, polygons...)
Poisson Kernel

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- Harder: $k \in C^{k,\alpha}$ iff $\partial\Omega \in C^{k+1,\alpha}$ (need $\alpha \in (0, 1)$, Jerison-Kenig!)

Kenig-Toro 90s, 00s: $\text{osc}\ k$ controls $\text{osc}\ \partial\Omega$. Vice Versa!

Hofmann-Martell & Azzam-Mourgoglou-Tolsa 2018: $k$ isn’t too small or too big too often ($A_\infty$-condition) iff $\partial\Omega$ looks flat at most points and scales (uniformly rectifiable).

Takeaway: Geometry of a set is characterized by solutions of Laplacian in complement of the set!
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Famous Open Q: What is the (maximal) dimension of the support of $\omega$?
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- Precise value completely open!
Higher Co-dimension

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Think: curve in $\mathbb{R}^3$. 
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Think: curve in $\mathbb{R}^3$. More exotic: snowflake in $\mathbb{R}^3$!
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Why do this? It is fun!
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Why do this? It is fun! Applications to Biology?

**Figure:** DNA Straightens and Curls up to Attract/Avoid Enzymes
Need degenerate elliptic PDE.
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\[ E = (x, \phi(x)) \subset \mathbb{R}^n. \phi : \mathbb{R}^d \to \mathbb{R}^{n-d}. \]
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**Question:** Geometry of \( E \) characterized by \( \omega_L \) vs \( \sigma \)?
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**Theorem (David-Feneuil-Mayboroda 2017)**

Let $E$ be the graph of a Lipschitz $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^{n-d}$ with small Lip constant. Then $\omega^X_{L_{\alpha}} = k^X \, d\sigma$ and $k^X$ is not too small or too big too often ($A_\infty$ weight).
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Note: applies to much more general scents (i.e. any suitably smooth replacement for \( D_\alpha(x)^{-(n-d-1)} \) \( I \) works).
What about the free boundary?

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General Free Boundary Problem:

Does the oscillation of \( k \) control the oscillation of \( \phi \)?

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If \( k = \) constant must it be that \( \phi = \) constant?

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**Theorem (David-E.-Mayboroda 18)**

For any $E, \alpha$ as above, have $\omega_{L\alpha} = \text{constant } d\sigma$. 

*Takeaway:* For magic $\alpha$, $d\omega_{\alpha}d\sigma$ doesn't control the regularity of $\phi$, and fails to do so in the most spectacular way possible!

*NOTE:* A version for when $E$ is fractal! $d$ non-integer (here $\omega_{L\alpha} \approx d\sigma$).

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$D_\alpha$ is too nice a scent!
What’s up with “magic $\alpha$”? 

Can compute: see that for $\alpha = n - d - 2$ we have

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When $\alpha$ is magic $D_\alpha(x) = \left( \int_E \frac{1}{|x-y|^{n-2}} d\sigma \right)^{-1/\alpha}$. Note: $\frac{1}{|x|^{n-2}}$ is harmonic!
Why is magic $\alpha$ magic?

- $D_\alpha$ satisfies an equation but what is really going on?
Open Questions about the Magic $\alpha$

1. Why is magic $\alpha$ magic?
   - $D_\alpha$ satisfies an equation but what is really going on?
   - Physical/geometric/probabilistic interpretation?

2. Is this emblematic or pathological?
   - Is any other $\beta$ magic?
   - Can we prove the converse for $\omega$ with $\beta$ not magic?

3. What does $\alpha \mapsto D_\alpha$ look like?
   - The power $-1$ makes this question harder.

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Thank You For Listening!

The way of Laplace!